Directions: Each quiz should be completed in 20 minutes. Please grade yourself harshly.

Quiz 1

1. Evaluate $\lim_{x\to 2}\frac{x^{10}-2^{10}}{x^5-2^5}$ [10 pts]

Solution: $\lim_{x\to 2}\frac{x^{10}-2^{10}}{x^5-2^5}=\lim_{x\to 2}\frac{(x-2)(x^9+2x^8+...+2^9)}{(x-2)(x^4-2x^4+...+2^4)}=\frac{10\cdot 2^9}{5\cdot 2^4}=2^6$

2. The position of a particle is given by $p(t)=t^2$. Calculate the velocity of the particle at t =1. [10 pts]

Solution: $p'(t)=2t$. Therefore the velocity is $p'(1)=2$

3. Suppose that *f(x)* is a bounded function that satisfies

$$1\leq f(x)\leq 5$$

Calculate $\lim_{x\to 0}x^2 f(x)$ [10 pts]

Solution: $x^2\leq x^2 f(x)\leq 5x^2$. Aplying the squeeze theorem yields
$0=\lim_{x\to 0}x^2\leq \lim_{x\to 0}x^2 f(x)\leq \lim_{x\to 0}5x^2=0$. Hence $\lim_{x\to 0}x^2 f(x)=0$.

Quiz 2

1. Compute $\lim_{x\to-\infty}\dfrac{\sqrt{5x^2-2}}{x+3}$ [10 pts]

Solution: $\lim_{x\to-\infty}\dfrac{\sqrt{5x^2-2}}{x+3}=\lim_{x\to-\infty}\dfrac{\sqrt{5x^2-2}/x}{x+3/x}=\lim_{x\to-\infty}\dfrac{-\sqrt{5-\frac{2}{x}}}{1+\frac{3}{x}}=-\sqrt{5}$

2. Let $f(x)=\dfrac{x^4-1}{x-1}$. For which x is *f(x)* discontinuous? Is the discontinuity(s) removable or not? [10 pts]

Solution: f (x) is a rational function and is therefore continuous everywhere where the denominator isn't 0. In particular, f (x) is not continuous at x = 1. Since $\lim_{x\to1}\dfrac{x^4-1}{x-1}=4$, the discontinuity is removable.

3. Suppose $\lim_{x\to\infty}f(x)=5$, what is $\lim_{x\to0^+}f\left(\dfrac{1}{x}\right)$? [10 pts]

Solution: $\lim_{x\to\infty}f(x)=5$ means that f (x) maps large values near 5. As x → 0^+, (1/x) → ∞.
Therefore $\lim_{x\to0^+}f\left(\dfrac{1}{x}\right)=5$.

Quiz 3

1. Let *f*: [0, 1] $\rightarrow$ (0, 1) be continuous. Show that for some x $\in$ [0, 1] $f(x) = x^2$
[10 pts]

Solution: By hypothesis, 0 < *f* (x) < 1. Consider the function $h(x) = f(x) - x^2$**.** This function is continuous, since *f* (x) is continuous. By hypothesis, 0 < *f* (x) < 1 and therefore $h(0) = f(0) - 0^2 > 0$, while $h(1) = f(1) - 1^2 < 1 - 1 = 0$. Hence the Intermediate-Value Theorem guarantees the existence of some number x such that $h(x) = 0$. But for this x, we must have $f(x) = x^2$.

2. Prove using a $\delta - \varepsilon$ argument that $\lim_{x \to -3}(2x+1) = -5$ [10 pts]

Solution: Notice that –5 = 2(-3) + 1. Therefore $|2x+1-(2(-3)+1)| = |2(x+3)|$. In particular, $|2x+1-(-5)| < \varepsilon$ whenever $0 < |x+3| < \delta(\varepsilon) = \dfrac{\varepsilon}{2}$

3. Prove using a $\delta - \varepsilon$ argument that $\lim_{x \to 1}(x^2 - 2x) = -1$ [10 pts]

Solution: Notice that –1 = $1^2 - 2(1)$. Therefore
$|x^2 - 2x - (1^2 - 2(1))| = |x^2 - 1 - 2(x-1)| = |x-1||x+1-2| = |x-1||x-1| = |x-1|^2$.
Theretofore $|x^2 - 2x + 1| < \varepsilon$ whenever $0 < |x-1| < \delta(\varepsilon) = \sqrt{\varepsilon}$.

Quiz 4

1. Let $f(x)=\begin{cases} x^2 Sin\left(\dfrac{1}{x}\right) & if \quad x\neq 0 \\ 0 & if \quad x=0 \end{cases}$

   (a) Determine whether $f'(0)$ exists. [5 pts]
   (b) Is *f* continuous at x = 0? How do you know? [5 pts]

Solution:

(a) $f'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{h^2 Sin\left(\frac{1}{h}\right)-0}{h}=\lim_{h\to 0} hSin\left(\dfrac{1}{h}\right)=0$, where the last limit has been computed with the squeeze theorem. Hence the derivative $f'(0)$ exists and equals 0.

(b) f is continuous at 0, because it is differentiable at 0. Recall that differentiability implies continuity, but not visa versa.

2. (a) Let $f(x)=x^{1/5}$. Use the definition of the derivative to compute $f'(x)$. [5 pts]
   (b) For what x is *f* differentiable? [5 pts]

Solution:

(a) $f'(x)=\lim_{y\to x}\dfrac{f(y)-f(x)}{y-x}=\lim_{y\to x}\dfrac{y^{1/5}-x^{1/5}}{y-x}=\lim_{y\to x}\dfrac{y^{1/5}-x^{1/5}}{(y^{1/5})^5-(x^{1/5})^5}$

$$=\lim_{y\to x}\frac{(y^{1/5}-x^{1/5})}{(y^{1/5}-x^{1/5})([y^{1/5}]^4+[y^{1/5}]^3[x^{1/5}]+...+[x^{1/5}]^4)}=\frac{1}{5[x^{1/5}]^4}=\frac{1}{5x^{4/5}}=\frac{1}{5}x^{1/5-1}$$

(b) The derivative exists provided x = 0.

3. Let $f(x)=2x^3-x+7$. Find the equation of the tangent line at the point x = 1.

[10 pts]

Solution: $f'(x)=6x^2-1$ so $f'(1)=5$. The equation $y-f(1)=f'(1)(x-1)$ identifies the line tangent to the curve at the point $(1, f(1))$. Therefore $y-8=5(x-1)$ is the desired equation.

4. Does the equation $\sqrt[3]{x}=1-x$ have a solution in (0, 1)? Justify your answer

[10 pts]

Solution: Set $f(x) = \sqrt[3]{x} - (1 - x)$ and observe that *f* is continuous on [0, 1]. Notice that $f(0) = -1 < 0$, while $f(1) = 1 > 0$. Therefore, by the Intermediate Value Theorem, $f(x) = 0$ for some x ∈ (0, 1). For this x, $\sqrt[3]{x} - (1 - x) = 0$ or, equivalently, $\sqrt[3]{x} = 1 - x$.